Complexity of the Game of Sprouts

Leemon Baird and Dino Schweitzer
Department of Computer Science, USAF Academy, CO

Abstract *- Sprouts is a simple, yet analytically interesting, game first developed in 1967 by Michael Paterson and John Conway. Mathematicians have analyzed the game for various strategies and mathematical properties. This paper adds to the body of knowledge by presenting a proof that several problems in the Game of Sprouts are NP-complete. In addition, for anyone wanting to conduct further research on the game, a complete annotated bibliography for Sprouts is listed.*

Keywords: computational complexity, mathematical games

1 Introduction

Sprouts is played by two players connecting spots with lines on a playing surface. The playing surface (piece of paper or computer screen) begins with *n* spots for the players to choose between. Players take turns connecting spots and adding a new spot along the drawn lines with the following constraints:

- The line must not touch or cross itself or any other line
- The new spot cannot be on an endpoint of the line, and thus splits the line into two parts
- No spot can have more than three lines connected to it; note that when a new spot is created, it starts with two lines already connected to it

Eventually, there are no legal moves remaining, and, in normal play, the player who makes the last move wins. In a variation of the game, misère play, the player who makes the last move loses. Figure 1 shows a simple game sequence with two starting spots. Player 1 connects the two spots creating a new spot in the middle. Player 2 connects the rightmost spot back with itself, creating a new spot, and removing the rightmost spot from further play since it has three lines connected to it. Play continues until the final move shown in the lower right game illustration. It is player 2's turn, but the only available spot already has two lines associated with it, so a line cannot be drawn from the spot back to itself, thus, player 2 has no move, and player 1 wins. The remaining live spot is called a *survivor*.

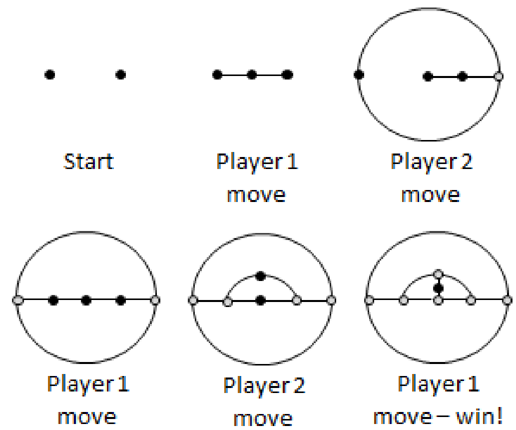
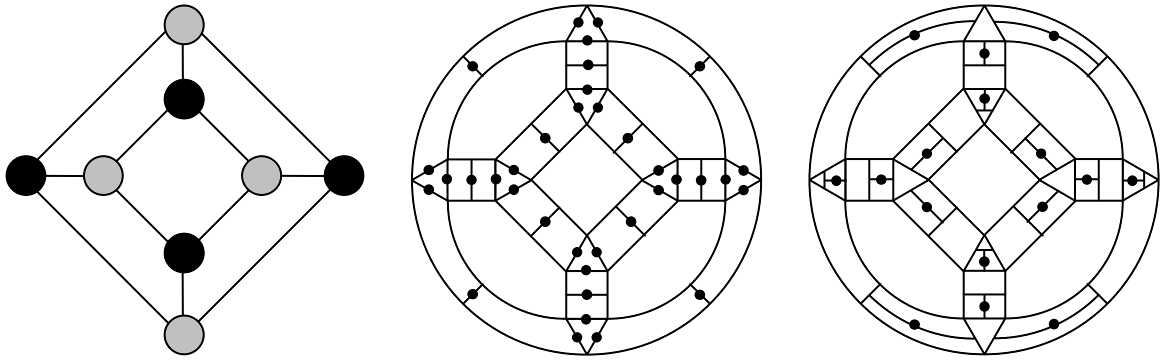


Figure 1. Five moves in a simple 2-spot Sprouts game.

2 Previous Work

A complete set of references to the literature for the Game of Sprouts is presented at the end of this paper. The earliest work dealt with the mathematics of finding optimal moves. Later work involved exhaustive search to solve the game to find optimal moves for certain starting boards, for both normal and misère play. Such searches on small games showed that the first player wins with a starting number of spots equal to 3, 4, or 5. The second player wins with spots equal to 1, 2, or 6. Lemoine and Viennot [17] extended this analysis to 32 spots. They conjectured that, in general, for arbitrarily-large numbers of starting spots, the first player has a winning strategy when the number of spots divided by six leaves a remainder of three, four, or five.

In addition to determining who has a winning strategy, other researchers have investigated the graph properties of the game, good layout of the game board by computer, and extending the game to surfaces other than a plane or sphere, such as a torus. Surprisingly, none of the previous analysis has documented the computational complexity of the game. This paper gives the first results along those lines, proving NP-completeness for questions about how long the game might last from a given starting position.



(a) (b) (c)

Figure 2. (a) a planar cubic graph with a maximal independent set of vertices colored gray. (b) a Sprouts position. (c) a Sprouts final position.

3 Proofs on Complexity

<u>Theorem 1</u>: The following problem is NP-complete: Given a Sprouts position and an integer k, can the game last more than k moves?

<u>Proof</u>: Proof is by reduction from the planar cubic graph maximum independent set problem. Given an integer k and a planar graph where every vertex is degree 3, the problem of determining if there is a set of at least k vertices with no two vertices adjacent, is known to be NP-complete. Figure 2a shows such a graph, Figure 2b shows a Sprouts position, and Figure 2c shows a possible final position in a game that started in position 2b.

The original graph in Figure 2(a) is converted into the Sprouts position in Figure 2(b). Each vertex in the original graph becomes one triangle in the Sprouts position with three live spots on the perimeter. Each arc in the original graph becomes a *connector* in the Sprouts position, consisting of two rectangular regions divided by a wall containing one live spot. The transformed graph in figure 2(b) shows only the live spots. There are dead spots at the other intersections.

In the figure, it may appear that there are intersections of degree 4, which is not possible in a valid Sprouts position. By shifting one line slightly at each degree-four intersection, it becomes a pair of degree-three intersections, so the graph actually represents a valid position. It is simply drawn with the degree-four intersections for clarity and simplicity.

If the original problem is whether there is a set of more than k independent vertices on an n-vertex graph, then the transformed problem is whether the Sprouts game can last more than $2n+k$. This is equivalent to asking whether a game can reduce the original $3n+3n/2$ live spots to under $n-k+3n/2$ live spots.

Note that each of the n vertices became a triangle, and that if we draw 2 lines within each triangle than the game will last $2n$ moves beyond the given board position. Such a game will end with no lines drawn within the connectors. It is also possible to draw only 1 line inside a given triangle and let the spot on its third side connect to the middle of the connector. In that case, the game still lasts $2n$ moves. However, if no lines at all are drawn inside the triangle, then lines can be drawn from its 3 live spots to the spots within all 3 of its adjacent connectors, and the game will extend for 1 extra move.

If the goal is to extend the game as long as possible, then as many triangles as possible should have lines drawn to all 3 of the incident connectors, and the rest of the triangles should simply have 2 lines drawn within the triangle. But a connector can only have lines drawn to at most one of the two triangles it connects. So the length of the game is maximized by choosing a maximal independent set of the nodes in the original graph, and allowing the corresponding triangles to connect to all 3 connectors. If there is an independent set of size k on the original graph, then the game can be made to last $2n+k$ moves in this way. Define MAX-SPROUTS as the problem of determining whether Sprouts can last at least k moves from a given position. Thus the maximal

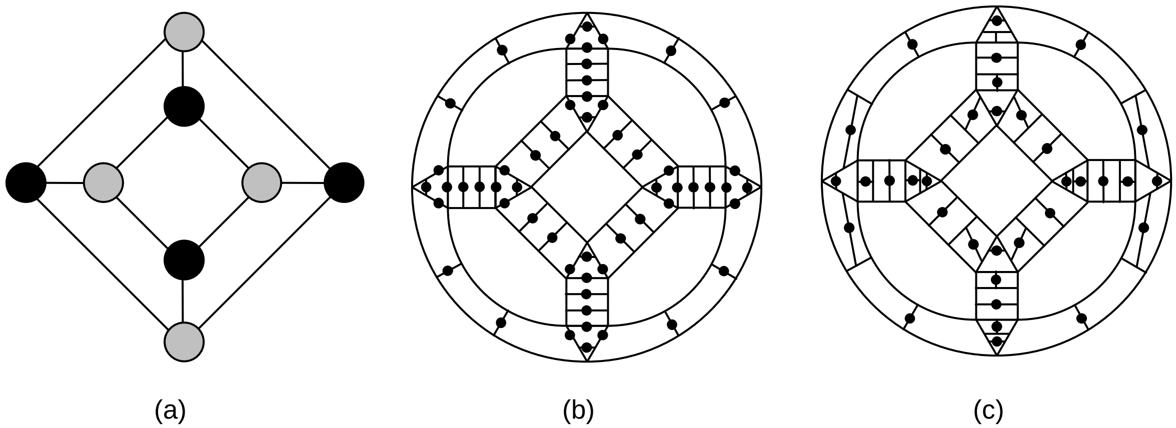


Figure 3. (a) a planar cubic graph with a maximal independent set of vertices colored gray. (b) a Sprouts position. (c) a Sprouts final position.

independent set problem on degree-3 planar graphs is reduced to MAX-SPROUTS, and so MAX-SPROUTS is NP-complete.

Theorem 2: The following problem is NP-complete: Given a Sprouts position and an integer *k*, can the game end in less than *k* moves?

Proof: Proof is by reduction from the same problem as in theorem 1, but the reduction is different.

In this case, the connectors are longer, and each triangle now has an additional spot within it. The game could proceed by drawing 3 lines within each triangle, and 1 in the middle of each connector, giving a total of $3n+3n/2$ moves. Or it go proceed by drawing no lines inside triangles, and just connecting each triangle to each of its 3 connectors, which would be a total of 3n moves. But the minimum number of moves is achieved by finding a maximal independent set on the original graph. For each vertex in the set in the original graph, draw just two lines inside the triangle. For each vertex not in the set, draw lines connecting its triangle to all 3 connectors, and draw no lines inside the triangle. If the maximal independent set had k vertices, then this game will last $2k+3(n\text{-k})$ moves. It's clear that there is no way to make the game any shorter than that. Define MIN-SPROUTS as the problem of determining whether the Sprouts game can finish in only *k* moves or fewer. The maximal independent set problem on degree-3 planar graphs is thus reduced to MIN-SPROUTS, and so MIN-SPROUTS is also NP-complete.

4 Sprouts References

4.1 Discussion Forums

- Sprouts-Theory
  http://groups.google.com/group/sprouts-theory

Discussions on the theory of Sprouts representations, analysis, etc.

- Geometry-Research (Math Forum)
  http://www.wgosa.org/swarthmore.htm

The above is a 1995 thread, which includes Conway describing a strategy for playing the game. The complete Math forum (1992-present) is at http://mathforum.org/kb/forum.jspa?forumID=130&start=0

4.2 Software

- GLOP
  http://sprouts.tuxfamily.org/wiki/doku.php?id=download

Program to search the game tree and find whether a given Sprouts position is a win for the first or second player. So far, it has determined the winner for all normal games of up to 32 spots (plus 34, 35, 40, 41, 47), and all misere games up to 17 spots.

- Aunt Beast and Small Beast are programs used by Sprouts competitors, but are not available on the Internet.

4.3 Websites

- University of Utah Sprouts Applet
  http://www.math.utah.edu/~alfeld/Sprouts/

This applet lets two humans play Sprouts. The computer does not play. The lines do not move.

- Game of Sprouts
  http://www.GameOfSprouts.com

Currently, the most complete listing available of Sprouts publications, software, and sites.

- World Game of Sprouts Association
  http://www.wgosa.org/

An organization that runs Sprouts tournaments, and discusses Sprouts strategies. (Was at http://www.geocities.com/chessdp/)

- SproutsWiki
  http://sprouts.tuxfamily.org/wiki/

The home page for GLOP, the program that has solved Sprouts for all normal games up to 32 spots, and all misere games up to 17 spots.

- Sprouts (game)
  http://en.wikipedia.org/wiki/Sprouts_(game)

Wikipedia article on Sprouts, with rules, theorems, pictures, and references.

- Dan Hoey's Game Notation
  http://www.ics.uci.edu/~eppstein/cgt/sprouts.html

- A December 2000 post
  geometry-research@mathforum.com

suggesting a Sprouts notation

- Madras College Brussels Sprouts
  http://www.madras.fife.sch.uk/maths/games/brusselsprouts.html

Short introduction with pictures.

- Madras College Sprouts
  http://www.madras.fife.sch.uk/maths/games/sprouts.html

Short introduction to Sprouts, with pictures and links.

- Pencil and Paper Games
  http://orion.math.iastate.edu:80/danwell/MathNight/ppg.html

List of rules and an animated image of a game.

- Sprouts
  http://compmath.wordpress.com/sprouts/

Short introduction with references.

4.4 Publications

[1] Anthony, Piers. 1969. Macroscope. New York: Avon.

[2] D. Applegate, G. Jacobson, and D. Sleator, Computer Analysis of Sprouts, Tech. Report CMU-CS-91-144, Carnegie Mellon University Computer Science Technical Report, 1991. Also in The Mathemagician and the Pied Puzzler (1999) honoring Martin Gardener; E. Burlekamp and T. Rogers, eds, A K Peters Natick, MA, pp. 199-201. http://www.cs.cmu.edu/~sleator/papers/sprouts.pdf

The classic paper on computationally solving Sprouts. Gives solutions up to 11 spots. Proposes the Sprouts Conjecture: the first player loses the n-spot game if n is 0, 1, or 2 modulo 6, and wins otherwise.

[3] Edwyn Berkelamp, John Conway, and Richard Guy, Winning Ways for your Mathematical Plays, vol. 2: Games in Particular, chapter 17, pp. 564-568, Academic Press, London, 1982.

[4] Edwyn Berkelamp, John Conway, and Richard Guy, Winning ways for your mathematical plays, vol. 1, 2001.

[5] Brown, Wayne & Baird, Leemon C. III (2008) "A graph drawing algorithm for the game of sprouts", The 2008 International Conference on Computer Graphics and Virtual Reality, Las Vegas, Nevada, July 14-17. http://leemon.com/papers/2008bb2.pdf

Describes algorithms for keeping curves spread out when drawing Sprouts positions

[6] Brown, Wayne & Baird, Leemon C. III (2008) "A non-trigonometric, pseudo area preserving, polyline smoothing algorithm", Journal of Computing Sciences in Colleges, (Also in the Proceedings of the Consortium for Computing Sciences in Colleges Mid-South Conference). http://leemon.com/papers/2008bb.pdf

Describes algorithms for making the curves smooth when drawing Sprouts positions.

[7] Butler, Ralph M., Trimble, Selden Y., Wilkerson, Ralph W., "A logic programming model of the game of sprouts",ACM SIGCSE Bulletin, Volume 19 , Issue 1 (February 1987), Pages: 319 – 323. http://portal.acm.org/citation.cfm?id=31726.31779

Describes how Sprouts positions are encoded so a 700-line Prolog program could play sprouts, making random moves.

[8] Copper, M. 1993. Graph theory and the game of sprouts. American Mathematical Monthly 100(May):478. http://compmath.files.wordpress.com/2008/08/graph_theory_and_the_game_of_sprouts.pdf

Proves constraints on the length of a game as a function of the number of initial spots, whether the final graph is connected, and whether the final graph is biconnected. Then proposes further questions, some of which were answered by Lam (A Math Monthly, 1997).

[9] Draeger, Joachim, Hahndel, Stefan, Koestler, Gerhard, and Rosmanith, Peter, Sprouts: Formalisierung eines topologischen spiels. Technical Report TUM-I9015, Technische Universitaet Muenchen, March 1990.

[10] Eddins, Susan Krulik, "Sprouts: Analyzing a simple game", IMSA Math Journal. http://staff.imsa.edu/math/journal/volume2/articles/Sprouts.pdf

A simple analysis with questions and answers, as for teaching students.

[11] Focardi, Riccardo and Luccio, Flamina, "A new analysis technique for the Sprouts Game", proc. of the 2nd International Conference on FUN with Algorithms 2001, Carleton Scientific Press, Isola d'Elba, Italy, May 2001. http://citeseer.csail.mit.edu/453452.html http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.21.212&rep=rep1&type=pdf

Describes a new way to analyze Sprouts positions, giving strategies based on isolating vertices, and gives the first formal proof for the game of 7 spots.

[12] Focardi, Riccardo and Luccio, Flaminia L., A modular approach to sprouts, Discrete Applied Mathematics 144 (2004), no. 3, 303–319. http://dx.doi.org/10.1016/j.dam.2003.11.008

From the abstract: We study some new topological properties of this game and we show their effectiveness by giving a complete analysis of the case $x_0=7$ for which, to the best of our knowledge, no formal proof has been previously given.

[13] Fraenkel, Aviezri S., "Combinatorial games: Selected biography with a succinct gourmet introduction" The Electronic Journal of Combinatorics 2009, #DS2. http://www.combinatorics.org/Surveys/ds2.ps

A survey of combinatorial games with a huge list of references, covering many related topics, including Sprouts.

[14] Gardner, Martin, Mathematical games: of sprouts and brussels sprouts, games with a topological flavor, Scientific American 217 (July 1967), 112–115.

[15] Gardner, Martin. 1989. Sprouts and Brussels sprouts. In Mathematical Carnival. Washington, D.C.: Mathematical Association of America.

[16] Lam, T.K. 1997. Connected sprouts. American Mathematical Monthly 104(February):116. http://compmath.files.wordpress.com/2008/08/connected_sprouts.doc

Answers questions posed by Copper (American Mathematical Monthly, 1993), about the graph that is obtained at the end of the game. Gives the length of the shortest game for connected and biconnected graphs.

[17] Lemoine, Julien and Viennot, Simon (2007) "A further computer analysis of Sprouts". http://download.tuxfamily.org/sprouts/sprouts-lemoine-viennot-070407.pdf (7 April 2007 version)

The first paper since 1991 giving the results for large games. It describes the authors' pseudo-canonization techniques, and gives their results for all games of up to 35 spots, except those with 27,30,31,32,33 spots. It also proposes the Nimber Conjecture: the nimber of the n-spot game is floor((n mod 6)/3).

[18] Lemoine, Julien and Viennot, Simon (2009) . "Sprouts games on Compact surfaces" http://arxiv.org/pdf/0812.0081v2 (2 March 2009 version).

Extends Sprouts theory beyond the usual plane/sphere case to other surfaces.

[19] Lemoine, Julien and Viennot, Simon (2009) . "Analysis of misere Sprouts game with reduced canonical trees" http://arxiv.org/pdf/0908.4407v1 (30 Aug 2009 version)

Describes how the authors were able to solve misere games up to 17 spots using reduced canonical trees, which are analogous to nimbers for normal Sprouts.

[20] Peterson, Ivar (1997) "Sprouts For Spring", Science News, April 5. http://www.sciencenews.org/sn_arc97/4_5_97/mathland.htm

Short Science News article from 5 April 1997 article giving the rules, history Macroscope quotes, and a few references.

[21] Pritchett, Gordon, "The game of Sprouts", Two-Year College Mathematics Journal, 7, 7 (Dec 1976) pp. 21-25. http://www.eric.ed.gov:80/ERICWebPortal/detail?accno=EJ152045

Proves several Sprouts theorems.