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# A Single-Sheet Icosahedral Folding With Improved Efficiency, Using a Business Card

Leemon Baird
Barry Fagin[1]
Department of Computer Science
2354 Fairchild Drive
US Air Force Academy
USAFA, CO 80840
719-333-3590

Max Fagin
Thayer School of Engineering
Dartmouth College
Maxwell.H.Fagin@dartmouth.edu

ABSTRACT

We show how to fold a business card into an icosahedron, using an origami construction (folding only, no cutting or rulers) with higher efficiency (40.8%) than any single-sheet folding we are aware of. Due to the aspect ratio of a standard business card, the resulting faces are just slightly non-equilateral, with an obtuse-to-acute angle ratio of 1.0085. This deviation is not detectable to an observer, but in the interest of completeness we also give a slightly more complex construction that results in a perfect icosahedron with a very slight reduction in efficiency.

## I. The Basic Construction

The standard American business card is 3.5"x 2", giving an aspect ratio of 7:4. Our construction works with any foldable rectangle with this aspect ratio, so for your first attempt you may wish to work with a large piece of cardboard of the correct shape.

Given an aspect ratio of 7:4, we may begin by folding the rectangle into sevenths in the following way. Unless explicitly indicated otherwise, all folds in the construction below are mountain folds.

First fold the lower left hand corner up to make a square, without creasing the diagonal. Mountain fold the right edge of the square:

Insert Figure 1 about here

[1] Please address all correspondence to this author, barry@faginfamily.net.

Fold the resulting square in half, then fold each half in half again:

Insert Figure 2 about here

Fold three more sections of the same width in the remaining section on the right, to give you a rectangle divided in sevenths:

Insert Figure 3 about here

Fold a diagonal crease from the upper left corner to the lower right:

Insert Figure 4 about here

Add creases through the points indicated below:

Insert Figure 5 about here

Add two more creases through the points indicated below:

Insert Figure 6 about here

Add the four new creases below:

Insert Figure 7 about here

Complete the folding process with the final 4 creases indicated here:

Insert Figure 8 about here

This completes all the mountain folds. Next, mark your folded paper as shown:

Insert Figure 9 about here

The dotted lines are the sites of valley folds to be performed in the steps to follow. With sufficient practice, these marks will not be necessary. Note that none of the lines or symbols will show up on the finished icosahedron. Only the shaded faces in the figure above will be visible in the final result.

Make the valley folds A,B,C,D shown below. The order does not matter.

Insert Figure 10 about here

Fold the squares together on valley fold pair E:

Insert Figure 11 about here

The final folds marked F below build the complete icosahedron. The squares near point E on the upper right form a slot for the circle near point X on the upper left, as do the squares near point E on the bottom left for the circle near point Y on the lower right. The mountain folds for the small corner triangles at the upper right and lower left lock the circles in. If you have performed this step correctly, the final combination of folds will bring the points X and Y to X' and Y'. Everything folds symmetrically.

Insert Figure 12 about here

## II. Deviations from Optimality

The construction above gives faces that deviate slightly from pure equilateral triangles. Consider our original construction:

Insert Figure 13 about here

Because all the non-diagonal folds are vertical, the angles labeled $\theta$ must be congruent to one another, as must the angles $\theta'$. Because they divide the rectangle into portions of equal width, we must have $90- \theta \equiv 90- \theta'$, which implies $\theta \equiv \theta'$. The remaining angle $\alpha$ is congruent to $180-2\theta$.

By the symmetry of the construction, the diagonal lines are parallel as well. Applying the geometry and the relationship between $\theta$ and $\alpha$ repeatedly, we see that all the non-right triangles are isosceles, constructed as follows:

Insert Figure 14 about here

Consider one of these triangles below:

Insert Figure 15 about here

Assume the original rectangle has total width B and height A. Then the triangle above will have height $H = A/(7/2)$, and width $W = B/7$. From above, $\theta = \tan^{-1}(2W/H) = \tan^{-1}( (2B/7) / (A/(7/2)) ) = \tan^{-1}(B/A)$ For our construction, $B/A = 7/4$, so $\theta = \tan^{-1}(7/4) \approx 60.255^{o}$, close to but not equilateral.

## III. An Exact Construction

Although the deviation from purely equilateral faces in the previous construction above is too small for an observer to detect, the resulting folds nonetheless do not produce a perfect icosahedron. In this section we provide a fully correct construction from a business card, at the cost of slightly increased complexity.

For equilateral faces, we require $\theta = \tan^{-1}(B/A) = 60^o$, which implies $B/A = \sqrt{3}$. This is slightly smaller than our original value of 7/4.

The hypotenuse of a right triangle with sides of length A and $\sqrt{3}A$ is of length 2A, so we may obtain a rectangle with the desired aspect ratio using two other identical rectangles in the following way:

Insert Figure 16 about here

The resulting rectangle has an aspect ratio of $\sqrt{3}A$ / A = $\sqrt{3}$, as required. For a business card, this results in the removal of $3½ - 2\sqrt{3} \approx .036$" inches off the length of the card.

Strict origami rules of paper folding do not allow cutting, in which case we simply mountain fold. This makes the triangles on the right edge slightly thicker, but does not affect the equilateral symmetry of the faces.

Once we have a rectangle of the required aspect ratio, we then require a way to fold it into sevenths. Our previous construction relied on an aspect ratio of 7:4, which is no longer the case. Instead, we may employ the method of "crossing diagonals", adapted from Lang [1], to find a point 1/7 along the long edge. The rectangle may then be accordion folded 3 times, with the remaining square divided into fourths as before.

## IV. Efficiency

For a card of width B and height A with triangles of width $W$ = B/7 and height $H$ = 2A/7, the area of each face in the first construction is $WH/2$ = AB/49. There are twenty faces in both constructions, and the area of the entire paper is AB. Thus the efficiency of the first construction is (20AB/49) / AB $\approx$ 40.816%.

For the second construction, assuming no cutting, the area of each face is $WH/2$ = AB'/49, and the area of the entire paper is AB, with B'/A = $\sqrt{3}$ and B/A = 7/4. Thus the efficiency is (20AB'/49AB) = 20B'/49B. B'/B = $(4\sqrt{3})/7$, so the efficiency is reduced by $(4\sqrt{3})/7$ below the previous construction, to $\approx$ 40.398%.

There are numerous examples of other icosahedral foldings, but almost all are modular, using multiple sheets of paper. Kasahara, for example [2], gives a construction from 20 separate sheets for both a stellate and a regular icosahedron. In [3], he gives a different construction for a tri-colored design using 30 modules. These are just the more well-known examples. A Google search on "modular origami polyhedra" produces several thousand hits.

Single-sheet foldings of polyhedra, and icosahedra in particular, appear to be considerably rarer. We have discovered only two, both with lower efficiencies than the one we have presented. Montroll [4], gives an origami icosahedral folding from a single square. Haga, in [5] also gives a single square folding. Neither provide efficiency calculations, so we provide our analyses here.

Below is the crease pattern for Montroll's construction:

Insert Figure 17 about here

Measuring from top to bottom along the diagonal, we have

$$S + 6H + S = L\sqrt{2}$$

Since the triangles are equilateral, substituting $H = \sqrt{3}S/2$ and simplifying gives

$$S = L\sqrt{2} / (3\sqrt{3} + 2)$$

The area of one face is $SH/2 = \sqrt{3}S^2/4$, so the area of the folded icosahderon is $5\sqrt{3}S^2$. Applying the formula above and simplifying, we have

$$A = (10\sqrt{3}/(3\sqrt{3} + 2)^2)\ L^2,$$ giving an efficiency of about 33.4%.

The crease pattern for Haga's construction is below:

Insert Figure 18 about here

From the relationship between the base and height of an equilateral triangle, and applying the Pythagorean theorem to the indicated right triangle, we have

$$2H = 2\sqrt{3}S/2 = L/2\sqrt{2} \rightarrow S = L/ 2\sqrt{6}$$

As before, the area of the folded icosahedron is $5\sqrt{3}S^2 = 5\sqrt{3}\ (1/24)\ L^2$, giving an efficiency of $5\sqrt{3}/24 \approx 36.1\%$.

## V. A Final Puzzle

We invite the reader to fold an icosahedron from this 7:4 rectangle, to see if the ink blots below form a recognizable pattern. It would certainly make an interesting business card!

Insert Figure 19 about here

NOTE: In some cultures, it is considered insulting to fold or damage someone else's business card. When in doubt, use your personal cards only!

## REFERENCES

[1] R. Lang, "Origami and Geometric Constructions", © 1996-2003, www.langorigami.com, page 14.

[2] K. Kasahara, *Origami Omnibus*, Japan Publications Inc, © 1988, ISBN 4817090014, pp 234-235

[3] K. Kasahara, *Amazing Origami,* Sterling Publishing Company, © 2001, ISBN 0806958219

[4] J. Montroll, *A Plethora of Polyhedra in Origami*, Dover Publications Inc, © 2002, ISBN 0486422712

[5] K. Kasahara and T. Takahama, *Origami for the Connoisseur,* Japan Publications Inc, © 1987, ISBN 4817090022, pp 62-63

FIGURES (color reproduction not required)

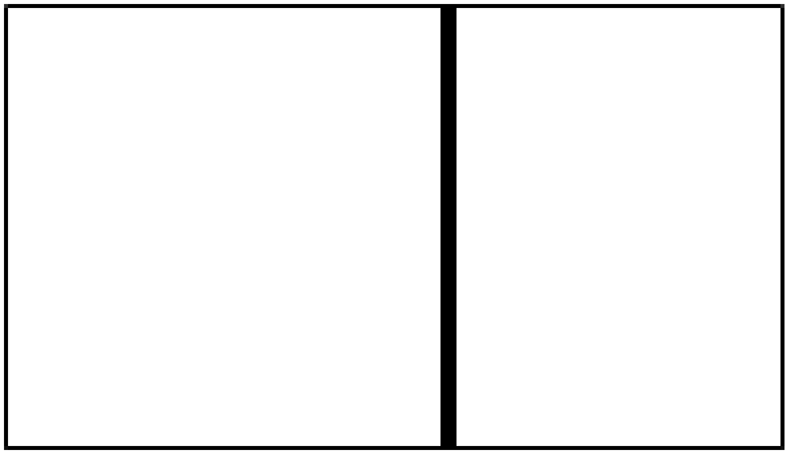


Figure 1

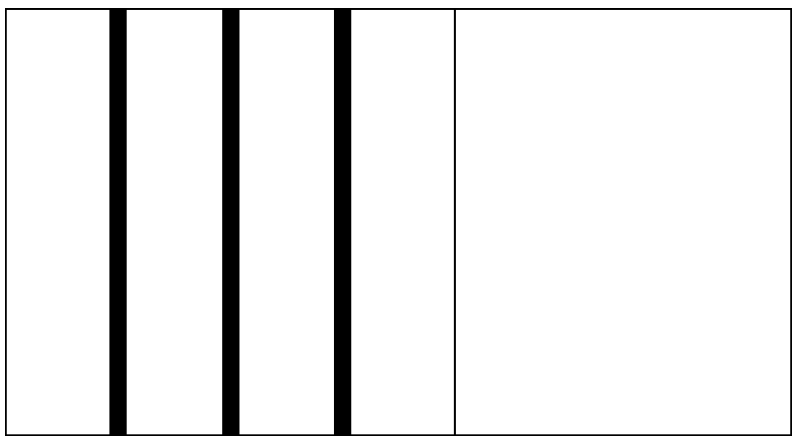


Figure 2

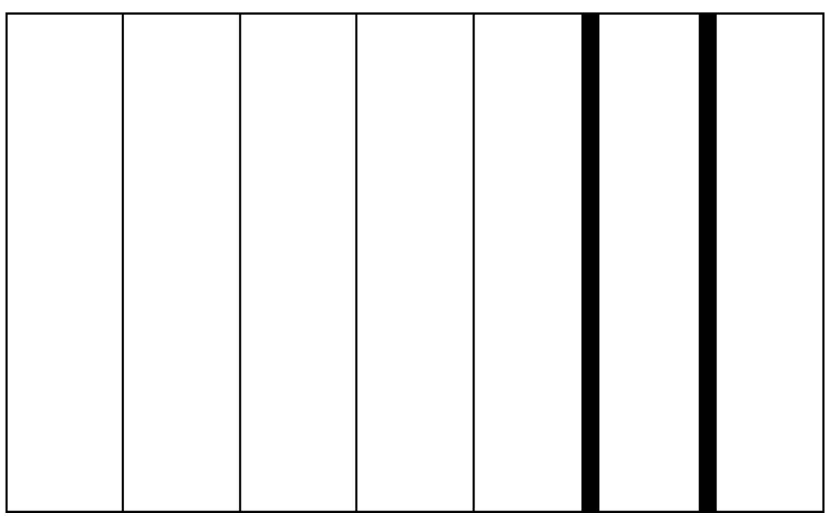


Figure 3

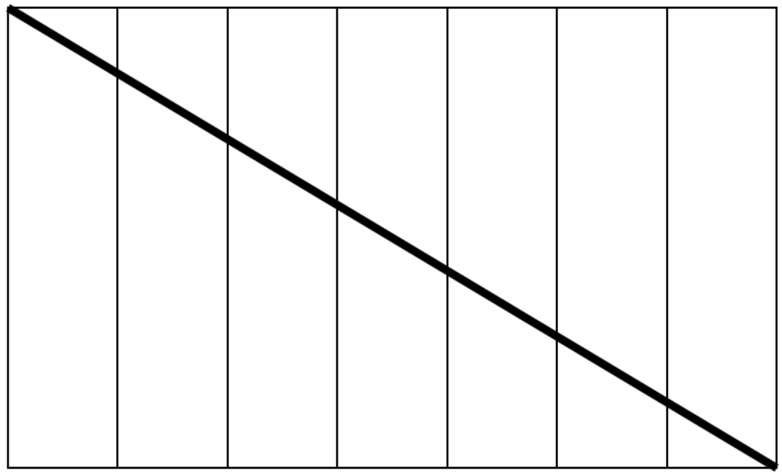


Figure 4

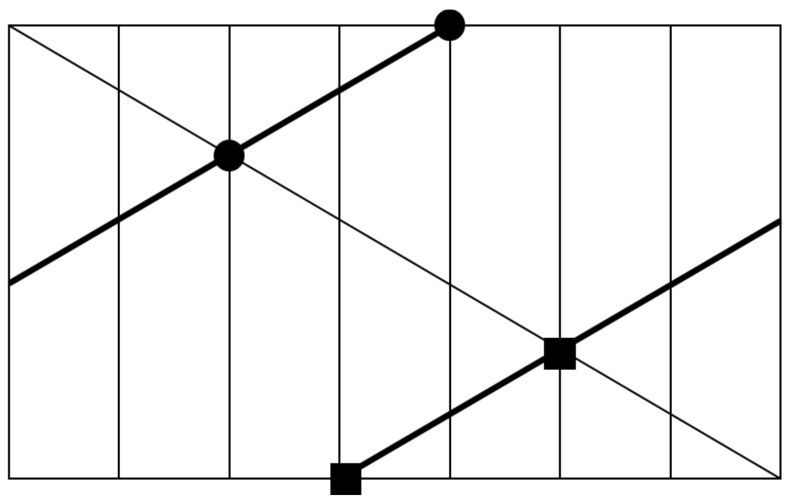


Figure 5

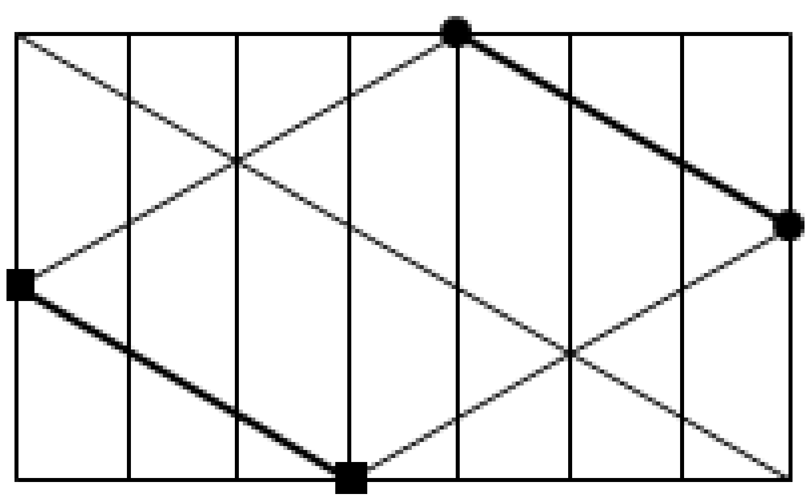


Figure 6

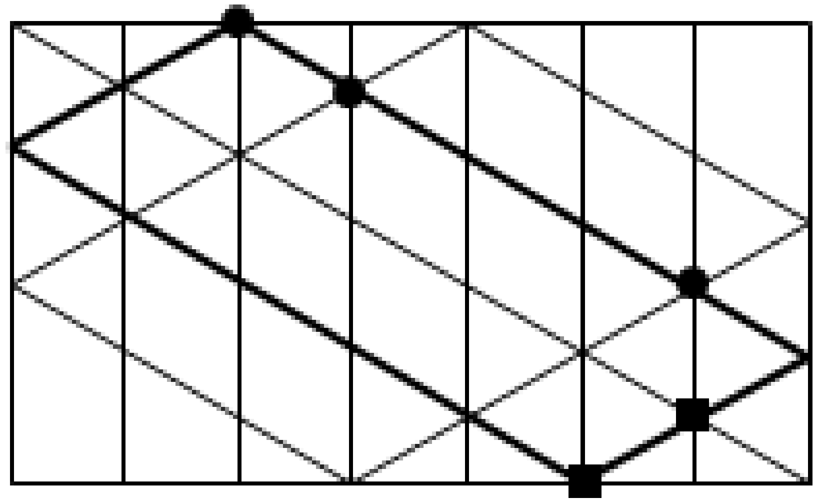


Figure 7

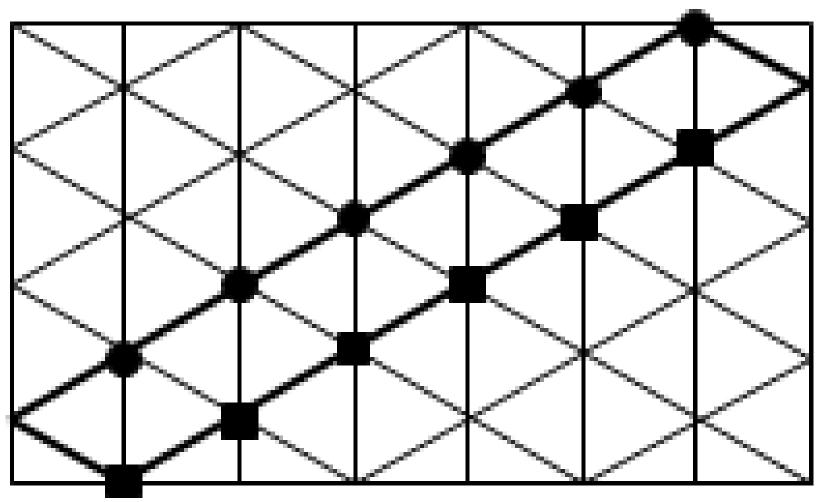


Figure 8

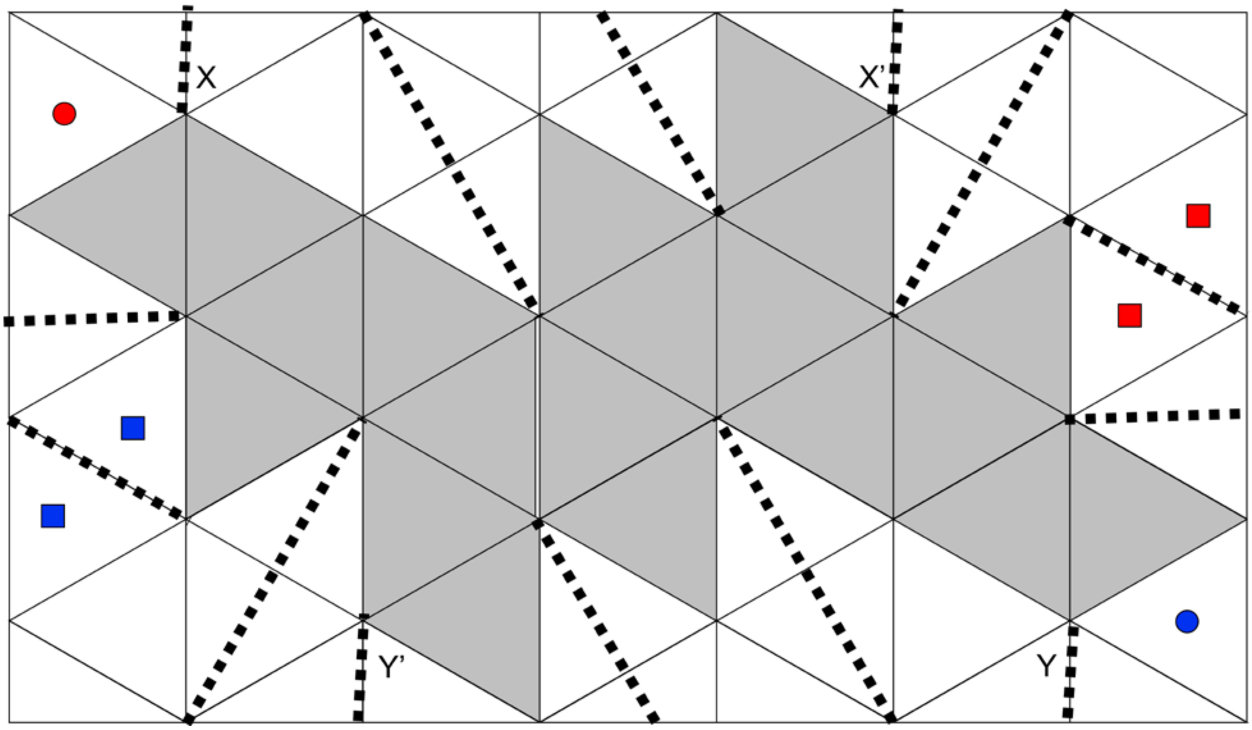


Figure 9

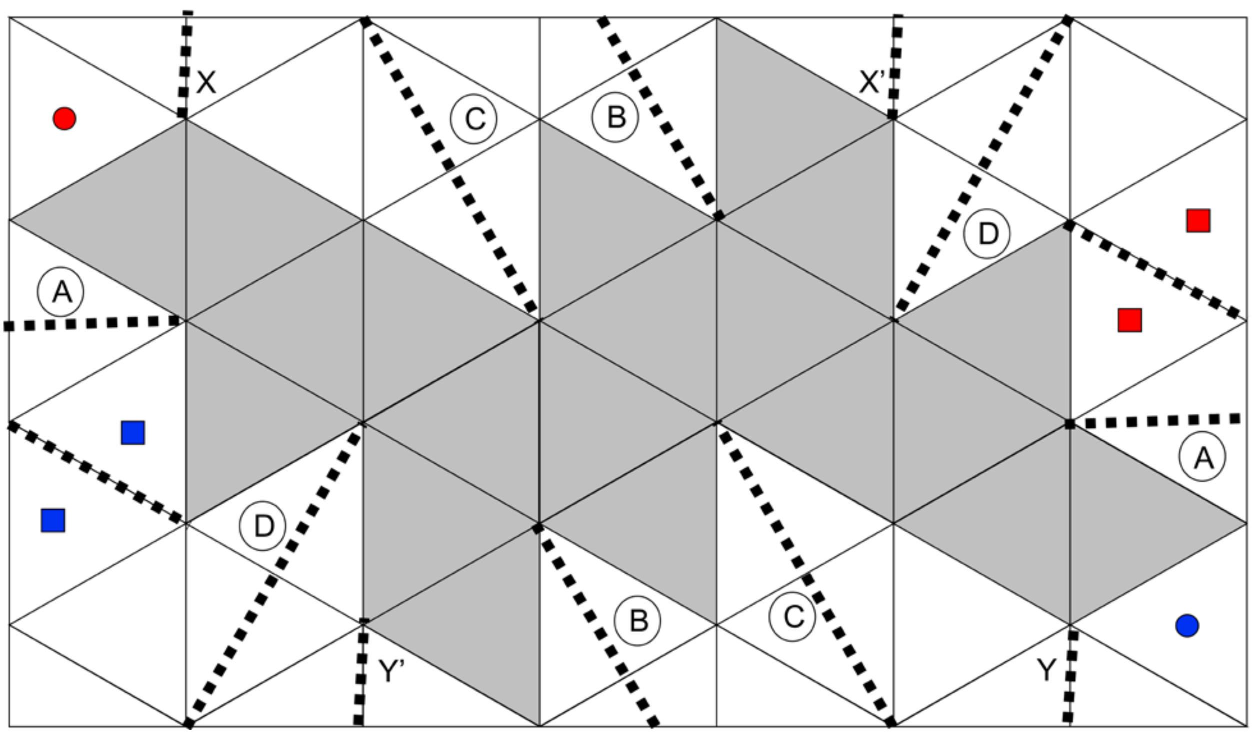
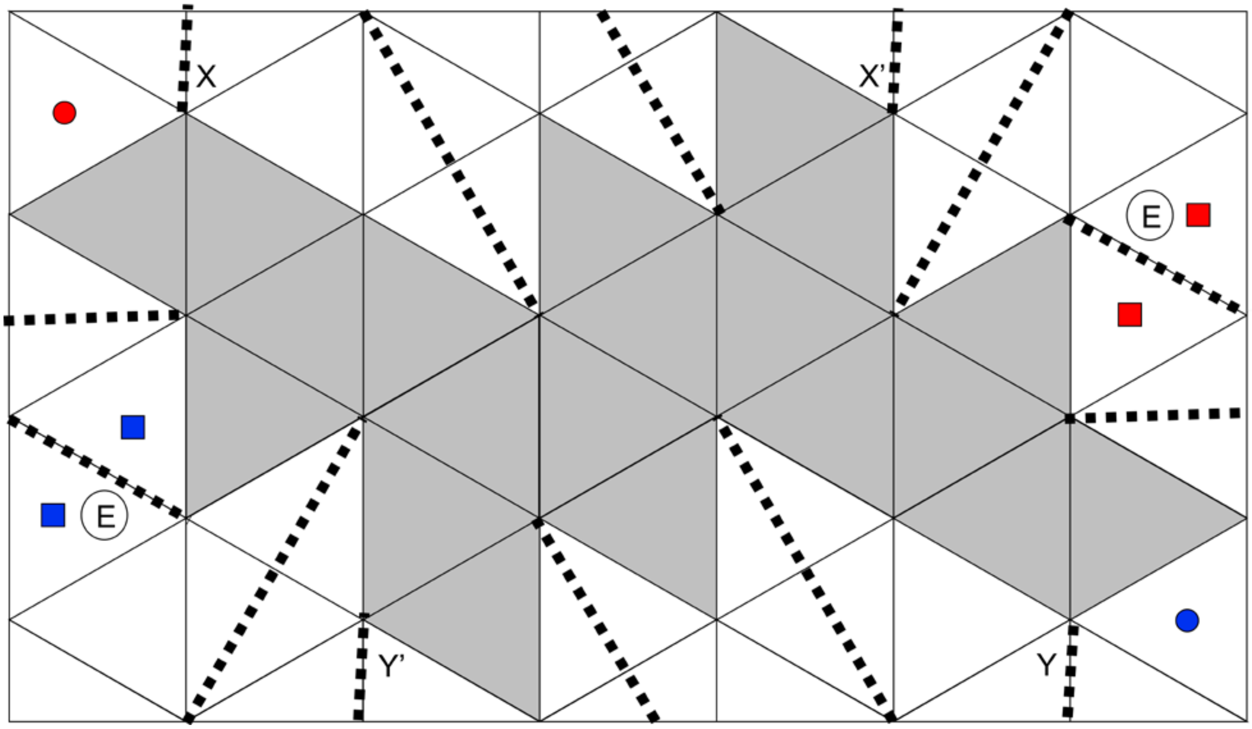


Figure 10



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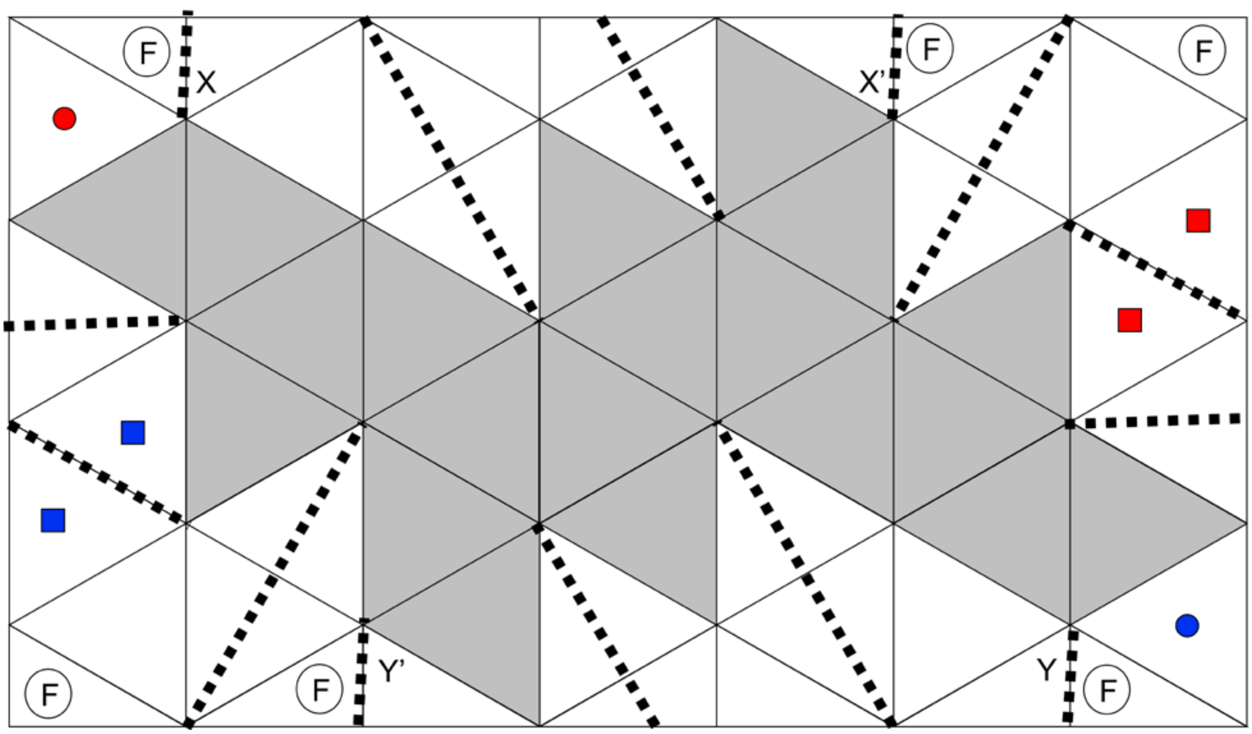

Figure 12

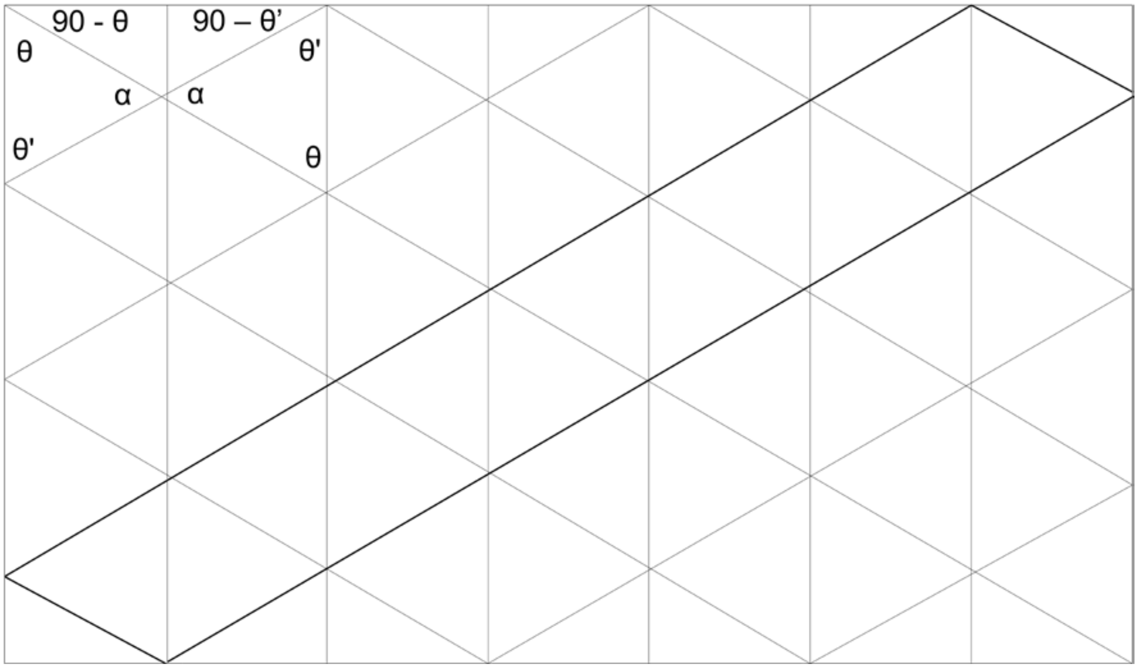


Figure 13

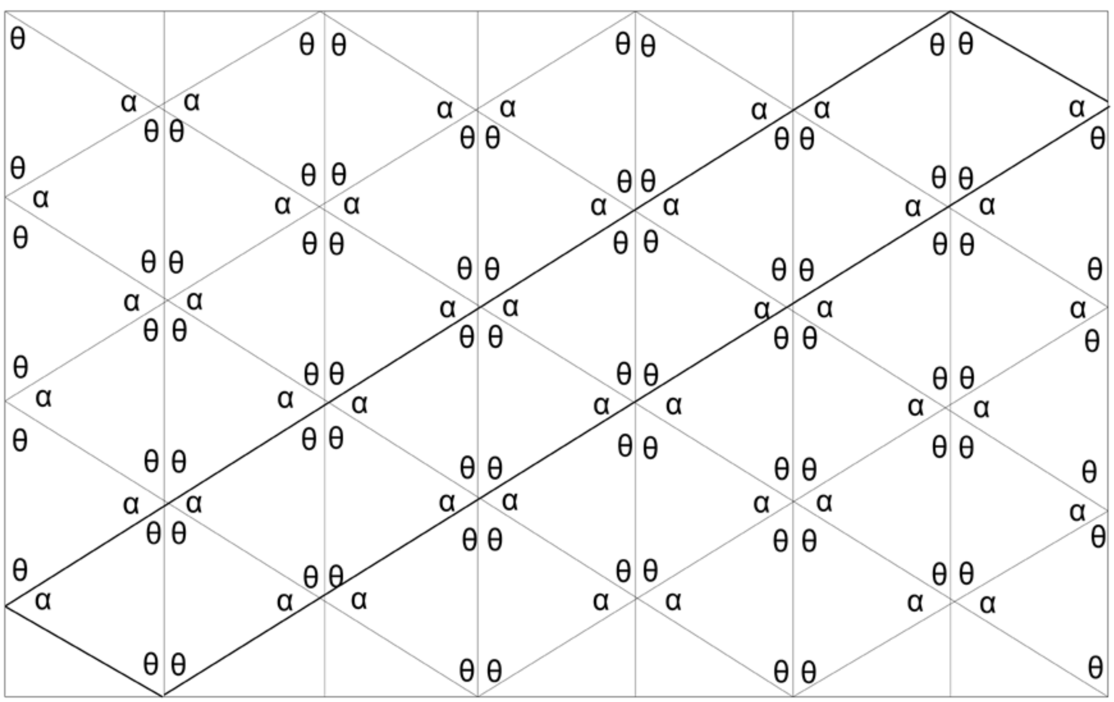


Figure 14

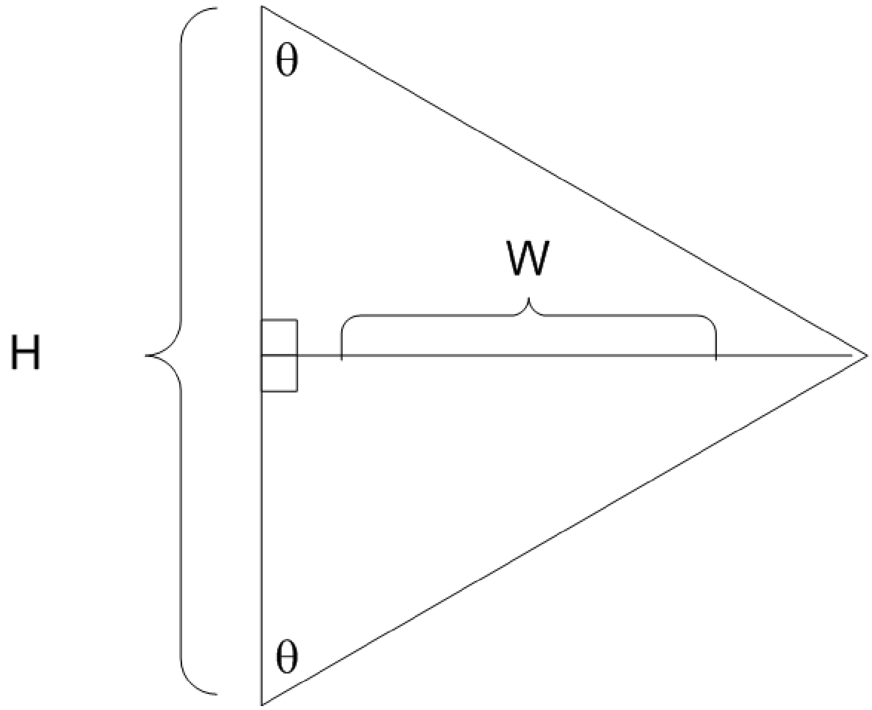


Figure 15

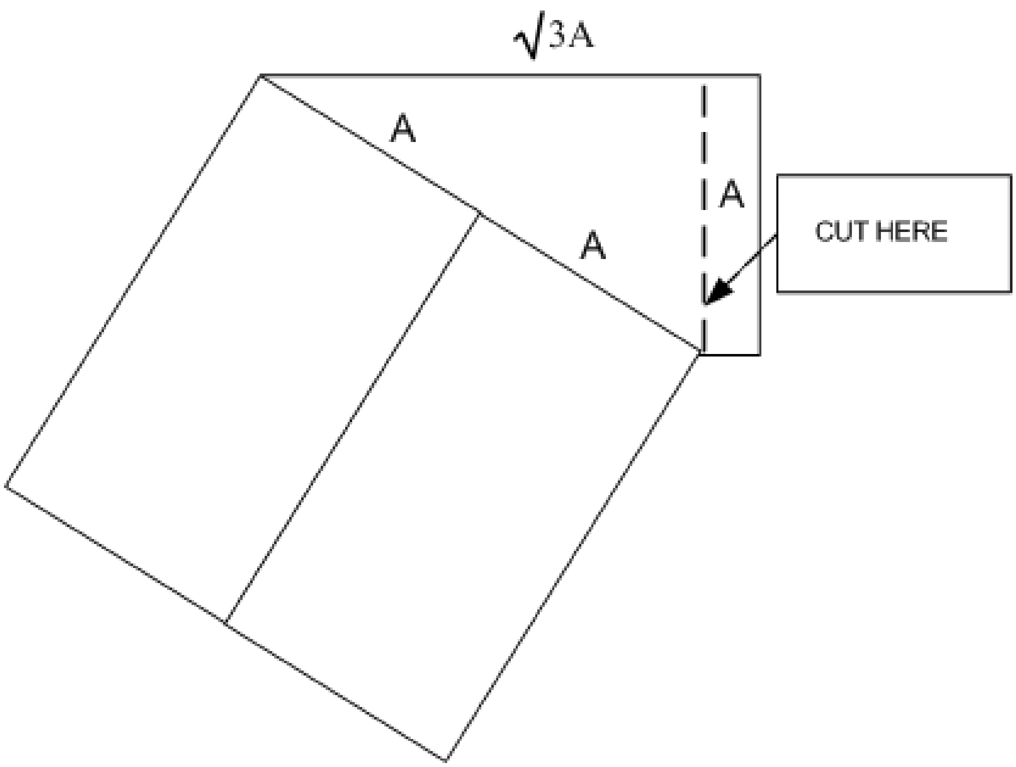


Figure 16

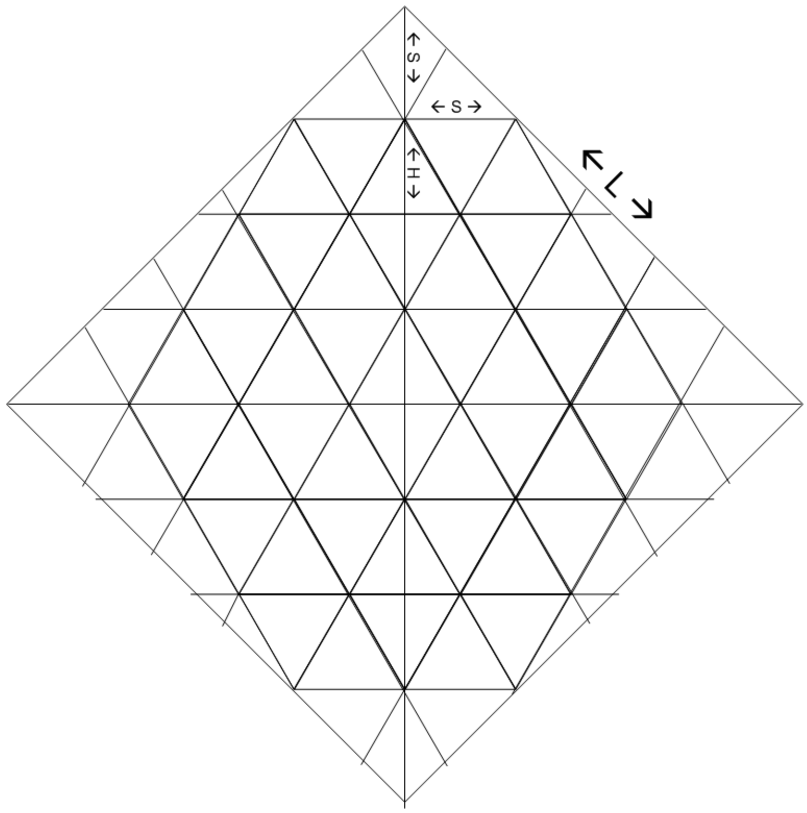


Figure 17: Montroll's construction

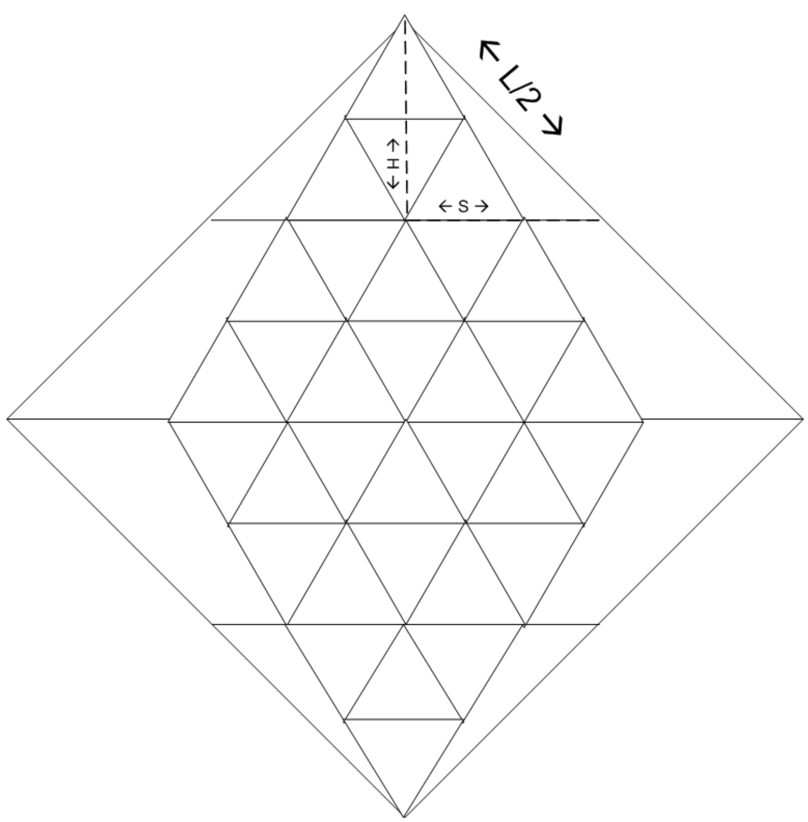


Figure 18: Haga's construction


Figure 19